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Flux and voltage periodic behaviour in a superconducting weak-link ring

J F Ralph†, T D Clark†, J Diggins†, R J Prance†, H Prance† and A Widom‡

† Physical Electronics Group, School of Engineering, University of Sussex, Falmer BN1 9QT, UK

‡ Department of Physics, Northeastern University, Boston, MA 02115, USA

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Abstract. Starting with a simple Lagrangian for the electromagnetic field with broken gauge symmetry, we derive an effective circuit Hamiltonian for a superconducting weak-link ring. The energy eigenstates of this Hamiltonian exhibit sensitivity to both external magnetic flux and an applied Faraday law voltage. We show that the flux periodicity of the rf SQUID ring, and the voltage periodic behaviour found in ultra-small-capacitance weak-link rings, can be seen as limiting cases of a more general phenomenon.

The operation of a radio frequency (rf) SQUID ring (a thick superconducting ring, containing a single Josephson weak-link device [1]) as a very sensitive magnetometer is well understood [2]. The response of the weak-link ring is periodic in an applied magnetic flux, with period $\Phi_0 = h/2e \simeq 2 \times 10^{-15}$ Wb. It is the small size of this periodicity which gives SQUID magnetometers their sensitivity as magnetic field sensors. However, it has also been shown that some ultra-small-capacitance weak-link ring structures can show voltage periodic behaviour [3] and be insensitive to a (static) applied magnetic flux. (This voltage dependent behaviour is often termed the 'charge mode', to distinguish it from the flux dependent 'flux mode' behaviour [4].) This periodic voltage dependence is usually attributed to the appearance of localized charge states around the weak link [5, 6], of the type which can also occur in singly connected Josephson junctions [7, 8] and Josephson arrays [9]. Such behaviour cannot be generated from the simple classical models [2, 10] or the macroscopic quantum Hamiltonians generally used to describe (flux mode) weak-link rings [11, 12]. Instead, an alternative model needs to be constructed [13], often using localized charge states as a basis [4]. The existence of the two types of behaviour, and the two types of theoretical model, has often led to confusion over the imposition of the appropriate commutation relation for a given system [14] (although this point has been discussed elsewhere [13]).

In this paper, we use a quantum mechanical model for the superconducting condensate in superconducting weak-link rings [15] which can be used to obtain both voltage periodic response and the flux periodic behaviour. Taking a simple Hamiltonian for the electromagnetic field, where the U(1) gauge symmetry is spontaneously broken, we construct an effective circuit Hamiltonian for a superconducting weak-link ring in terms of local discrete-field operators [15]. We then impose the usual canonical commutation relations

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between the local operators and their conjugate momenta. The behaviour of this system is discussed in the presence of an applied magnetic flux and an applied Faraday law voltage (i.e. a time dependent magnetic flux). We demonstrate that a single model can be used to obtain the usual (flux periodic) Hamiltonian for an rf SQUID ring, and voltage periodic (charge mode) behaviour, associated with localized charge states. In this model, we find that the two types of behaviour found in experimental systems may be limiting cases of a more general phenomenon. We concentrate on the behaviour of the condensate degrees of freedom, and do not discuss the behaviour of the (orthogonal) quasi-particle/normal fluid degrees of freedom which are normally associated with dissipative processes [8]. However, we will discuss the possible effect of general environmental degrees of freedom on the condensate, and consider the transition from one mode to the other in an experimental system [16].

The appearance of the superconducting condensate is associated with a broken gauge symmetry [17, 18]. Inside the superconductor, the usual U(1) gauge symmetry is spontaneously broken to $\mathcal{Z}_2$. Treating the electromagnetic fields as classical fields, and solving the resultant field equations, can reproduce many of the features normally associated with bulk superconductors [17]. However, this classical approach neglects the fact that the fields are quantum electrodynamic in origin and should still obey the correct canonical commutation relations [13, 19], albeit in a symmetry broken case. Taking a simple Lagrangian density and imposing the correct constraints [20], in the Coulomb gauge ($\nabla \cdot \boldsymbol{A} = 0$), we obtain a Hamiltonian density of the form [15]

$$\mathcal{H} = \frac{\varepsilon_0 \boldsymbol{E}^{\perp 2}}{2} + \frac{\mu_0^{-1}}{2}(\boldsymbol{\nabla} \times \boldsymbol{A})^2 - \frac{\rho \, \triangle^{-1} \rho}{2\varepsilon_0} + \frac{1}{2}\left(\frac{\hbar^2}{2\nu q^2}\right)\frac{\rho^2}{\mathcal{N}} + \frac{\mathcal{N}\gamma q^2}{\hbar^2}(\boldsymbol{\nabla}\phi - \boldsymbol{A})^2 \tag{1}$$

where $q = 2e$, $\boldsymbol{E}^{\perp} = -\partial \boldsymbol{A}/\partial t$ is the transverse part of the electric field, $\triangle \equiv \nabla^2$ and $\rho = -(2\mathcal{N}\nu q^2/\hbar^2)(\partial\phi/\partial t + A^0)$ is the real (Gaussian) charge density. This Hamiltonian density can be obtained from the time dependent Ginzburg–Landau model [21, 22], expressing the complex order parameter as $\Psi = \sqrt{\mathcal{N}}\exp(\mathrm{i}q\phi/\hbar)$ and taking the condensate density $\mathcal{N}$ to be a smooth, static field (i.e. ignoring the derivative terms for $\mathcal{N}$) [15]. (The number density is often associated with topological excitations (vortices) [21] which are difficult to describe with a quantum mechanical Hamiltonian.) Comparison with the Ginzburg–Landau model gives $\gamma = \hbar^2/2m_e$ and $\nu = 3\gamma/v_F^2$ (where v_F is the Fermi velocity in the non-superconducting material) [22]. For low electric and magnetic fields, the number density is approximated by $\mathcal{N} \simeq (a/b)$ where a and b are the usual Ginzburg–Landau coefficients [21].

The general form of the Hamiltonian comes from the broken gauge symmetry [17], but the values of the parameters (γ, ν and $\mathcal{N}$) are motivated by comparison with the time dependent Ginzburg–Landau theory. This, in turn, can be derived from the BCS theory, where the order parameter is proportional to the energy gap function [23]. It is applicable far below the critical temperature, where the energy gap function can be shown to obey a wave equation [24], and holds as long as fluctuations are over length scales which are long compared to the coherence length ξ and on time scales longer than $\sqrt{3}\xi/v_F$ [25].

The canonical commutation relations for this Hamiltonian are given by [13, 15]

$$[E_i^{\perp}(x), A_j(y)] = \frac{\mathrm{i}\hbar}{\varepsilon_0}\Pi_{ij}\delta(x-y) \tag{2}$$

where $\Pi_{ij} = \delta_{ij} - \triangle^{-1}\,\partial_i\,\partial_j$, and

$$[\rho(x), \partial_i\phi(y)] = \mathrm{i}\hbar\,\partial_i\delta(x-y). \tag{3}$$

In the absence of broken gauge symmetry, the electromagnetic field has two physical fields (the two polarizations of the transverse photons). The scalar and longitudinal photons are said to be unphysical. Their dynamical effects cancel out, leaving only the Coulomb contribution to the total energy [19]. This is also true for a normal conductor, even if it has an infinite conductivity [17].

In the superconductor, however, we can see that Hamiltonian (1) contains three dynamical fields. In addition to the transverse fields, the ϕ-field (the Nambu–Goldstone field [17]) also has physical excitations, related to the longitudinal and scalar photons by the fact that its conjugate momentum is the real (Gaussian) charge density ρ. It has already been noted that this additional field gives rise to a non-geometrical capacitance (the fourth term in (1)) [15], and that this self-capacitance can dominate the behaviour of some model SQUID rings [26]. The fact that such behaviour cannot occur in a normal conductor means that conventional electrical circuit models for superconducting circuits must be treated with extreme care.

Given a particular circuit, and assuming that the approximations used to obtain (1) are appropriate, we should be able to use this Hamiltonian to solve for the behaviour of the system. To solve the full-field Hamiltonian for the circuit modes (longitudinal and transverse) of a particular circuit would be an enormous task. Instead, we confine ourselves to discussing a few low-energy modes, which we can represent in terms of effective circuit elements and local discrete field operators.

The circuit that we wish to consider is a thick (compared to the London penetration depth [21]) niobium ring, containing a Josephson point contact weak link [2]. Well below the critical temperature (T_c) and the lower critical field (H_{c1}) (and subject to the other conditions discussed above), the superconducting condensate in niobium should be reasonably well described by the Hamiltonian density (1) with purely local interactions. Below H_{c1}, niobium (a type II superconductor) will be in the Meissner state, and whatever vortices are present should not influence the dynamics of the system. Other materials, particularly ones which are better described by Pippard's equation [21], will generally require the inclusion of non-local interactions [17], but these are not considered here.

We divide the ring up into three segments: two segments for the point contact ($n = 1, 2$) and one for the bulk ring ($n = 3$). We require at least two regions of weak superconductivity to allow for spatial localization of the real (Gaussian) charge (see below). The fields in each segment will be described by a transverse electric flux (Q_n), a magnetic flux (Φ_n), a Gaussian charge (q_n) and a phase difference ($\Delta\phi_n$), defined by [13, 15]

$$Q_n = \varepsilon_0 \int_{S_{Q_n}} \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{E}^{\perp}(x) \tag{4}$$

$$\Phi_n = \int_{S_{\Phi_n}} \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{B}(x) = \int_{\partial S_{Q_n}} \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{A}(x) \tag{5}$$

$$q_n = \int_{V_n} \mathrm{d}V \rho(x) \tag{6}$$

$$\Delta\phi_n = \int_{c_n} \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{\nabla}\phi(x) \tag{7}$$

where the open surfaces (S_{Q_n} and S_{Φ_n}), volumes (V_n) and curves (c_n) are defined in figure 1. The Gaussian charge is quantized in units of $q = 2e$ (the charge of one Cooper pair [21]), so that the charge operator can be represented by $q_n = 2eN_n$, where N_n is a number operator, and $\Delta\phi_1$ is an angular operator (period $= \Phi_0$) [15]. The electric flux operator Q_n, whilst it does have the units of charge, is a continuous operator and does not correspond to a

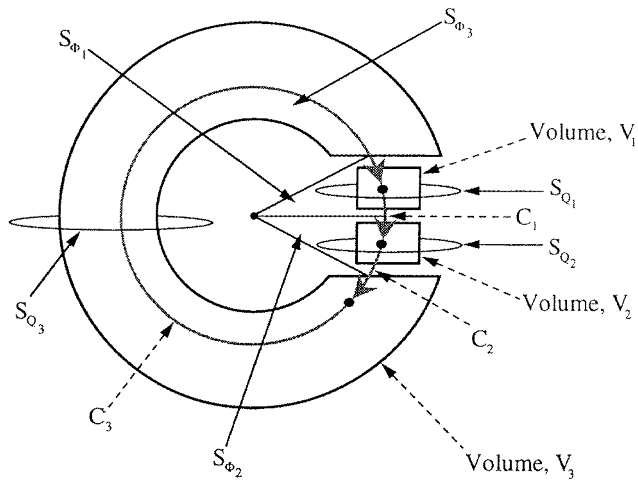

Figure 1. Schematic diagram of a weak-link ring, with two small regions of weak superconductivity ($n = 1, 2$) and one large bulk ring ($n = 3$).

distribution of real charge (it is sourceless). The only non-zero commutation relations are

$$[Q_n, \Phi_m] = \mathrm{i}\hbar\delta_{nm}$$

$$[q_n, \Delta\phi_m] = \begin{cases} \mathrm{i}\hbar & \text{if } m = n \\ -\mathrm{i}\hbar & \text{if } m = (n-1) \\ 0 & \text{otherwise.} \end{cases}$$

The resultant Hamiltonian is given by

$$\begin{aligned} H = {} & \left(\frac{Q_1^2}{2C_1^{(t)}} + \frac{\Phi_1^2}{2\Lambda_1}\right) + \left(\frac{Q_2^2}{2C_2^{(t)}} + \frac{\Phi_2^2}{2\Lambda_2}\right) + \left(\frac{Q_3^2}{2C_3^{(t)}} + \frac{\Phi_3^2}{2\Lambda_3}\right) + \frac{q_2^2}{2C_{22}} + \frac{q_2 q_3}{2C_{23}} + \frac{q_3^2}{2C_{33}} \\ & + \frac{\hbar I_{c_1}}{q}\left[1 - \cos\left(\frac{2\pi\,\Delta\phi_1}{\Phi_0}\right) - \left(\frac{q\Phi_1}{\hbar}\right)\sin\left(\frac{2\pi\,\Delta\phi_1}{\Phi_0}\right) + \frac{q^2\Phi_1^2}{2\hbar^2}\right] \\ & + \frac{qI_{c_3}}{2\hbar}(\Delta\phi_3 - \Phi_3)^2 + \frac{\hbar I_{c_2}}{q}\left[1 - \cos\left(\frac{2\pi(\Delta\phi_1 + \Delta\phi_3)}{\Phi_0}\right)\right. \\ & \left. - \left(\frac{q\Phi_2}{\hbar}\right)\sin\left(\frac{2\pi(\Delta\phi_1 + \Delta\phi_3)}{\Phi_0}\right) + \frac{q^2\Phi_2^2}{2\hbar^2}\right] \end{aligned} \tag{8}$$

where the transverse field energies have been parametrized in terms of effective inductances (Λ_n) and transverse capacitances ($C_n^{(t)}$). We have also used the fact that the total charge contained in the ring and the total phase change around the ring are constant,

$$q_1 + q_2 + q_3 = 0$$

$$\Delta\phi_1 + \Delta\phi_2 + \Delta\phi_3 = m\Phi_0$$

to remove q_1 and $\Delta\phi_2$ (the total charge is taken to be zero, and m is an integer). The capacitances for the real charge are combinations of the longitudinal (Coulomb) capacitance matrix elements $(C_{(l)})_{nm}$ and the condensate self-capacitances $C_n^{(sc)} = 3qI_c(\Delta x)^2/\hbar v_F^2$

(where the length of the segment is Δx), and are given by

$$\frac{1}{C_{22}} = (C^{-1}_{(l)})_{11} - (C^{-1}_{(l)})_{12} - (C^{-1}_{(l)})_{21} + (C^{-1}_{(l)})_{22} + \frac{1}{C^{(sc)}_1} + \frac{1}{C^{(sc)}_2}$$

$$\frac{1}{C_{23}} = 2(C^{-1}_{(l)})_{11} - (C^{-1}_{(l)})_{12} - (C^{-1}_{(l)})_{21} - (C^{-1}_{(l)})_{13} - (C^{-1}_{(l)})_{31} + (C^{-1}_{(l)})_{23} + (C^{-1}_{(l)})_{32} + \frac{2}{C^{(sc)}_1}$$

$$\frac{1}{C_{33}} = (C^{-1}_{(l)})_{11} - (C^{-1}_{(l)})_{13} - (C^{-1}_{(l)})_{31} + (C^{-1}_{(l)})_{33} + \frac{1}{C^{(sc)}_1} + \frac{1}{C^{(sc)}_3}.$$

Now we let the critical current for the bulk segment tend to infinity to obtain the constraint $\Delta\phi_3 \simeq \Phi_3$. To impose this constraint consistently, we must revert to the classical Hamiltonian, put $\Delta\phi_3 = \Phi_3$ and $\mathrm{d}(\Delta\phi_3)/\mathrm{d}t = \mathrm{d}\Phi_3/\mathrm{d}t$, and then re-quantize [26]. Doing this, we obtain

$$\begin{aligned} H = &\left(\frac{Q_1^2}{2C_1^{(t)}} + \frac{\Phi_1^2}{2\Lambda_1}\right) + \left(\frac{Q_2^2}{2C_2^{(t)}} + \frac{\Phi_2^2}{2\Lambda_2}\right) + \frac{\Phi_3^2}{2\Lambda_3} + \frac{q_2^2}{2C'_{22}} + \frac{q_2 Q'_3}{2C'_{23}} + \frac{(Q'_3)^2}{2C'_{33}} \\ &+ \frac{\hbar I_{c_1}}{q}\left[1 - \cos\left(\frac{2\pi\,\Delta\phi_1}{\Phi_0}\right) - \left(\frac{q\Phi_1}{\hbar}\right)\sin\left(\frac{2\pi\,\Delta\phi_1}{\Phi_0}\right) + \frac{q^2\Phi_1^2}{2\hbar^2}\right] \\ &+ \frac{\hbar I_{c_2}}{q}\left[1 - \cos\left(\frac{2\pi(\Delta\phi_1 + \Phi_3)}{\Phi_0}\right) - \left(\frac{q\Phi_2}{\hbar}\right)\sin\left(\frac{2\pi(\Delta\phi_1 + \Phi_3)}{\Phi_0}\right) + \frac{q^2\Phi_2^2}{2\hbar^2}\right] \end{aligned} \tag{9}$$

where $-Q'_3 = -q_3 - Q_3$ is the new conjugate momentum to Φ_3 (giving $[Q'_3, \Phi_3] = \mathrm{i}\hbar$) and is taken to be a continuous operator. The new capacitances are $C'_{33} = C_{33} + C_3^{(t)}$, $C'_{23} = C_{23}(1 + C_3^{(t)}/C_{33})$, and

$$C'_{22} = \left(\frac{1}{C_{22}} + \frac{C_3^{(t)} C_{33}}{4C_{23}^2(C_{33} + C_3^{(t)})}\right)^{-1}.$$

If we allow the critical current of one of the weak-link segments (I_{c_1} or I_{c_2}) to tend to infinity, the phase variable ($\Delta\phi_1$) will become localized and the Hamiltonian reduces to the usual macroscopic Hamiltonian for the rf SQUID ring [11, 12] and the system displays only flux periodic behaviour [26]. The introduction of two weak-link segments allows for the localization of charge, both in charge-phase space and in real space. However, the critical current of a real superconductor is not infinite. This approximation merely facilitates the introduction of the constraint $\Delta\phi_3 = \Phi_3$. A more rigorous approach would include the bulk segment and the weak-link segments on an equal footing (with longitudinal and transverse fields for each). This would allow localized charge states to exist in the bulk superconductor. Such excitations would violate the constraint imposed above but would tend to have a very high energy. Having restricted ourselves to a few, low-energy circuit modes, the additional simplification is valid.

We note that, in contrast to other models based on normal electrical circuits (with unbroken symmetry) [27], the two-weak-link Hamiltonian (9) will give energy states corresponding to physical excitations of the longitudinal and the transverse fields. The extra degree of freedom comes directly from the broken gauge symmetry. It is this longitudinal degree of freedom, together with the quantization of charge in units of $q = 2e$, which gives rise to voltage periodic behaviour.

Having taken the critical currents for the $n = 1$ and $n = 2$ segments to be relatively weak, we assume that Φ_1 and Φ_2 are fairly well defined and that we can treat the coupling between the phase across each weak link and the transverse fields surrounding it as negligible. If we also include an external (possibly time dependent) magnetic flux ($\Phi_{ex}(t)$) coupling to the bulk ring, we obtain

$$H = \frac{q_2^2}{2C'_{22}} + \frac{q_2 Q'_3}{2C'_{23}} + \frac{(Q'_3)^2}{2C'_{33}} + \frac{(\Phi_3 - \Phi_{ex}(t))^2}{2\Lambda_3} - \frac{\hbar I_{c_1}}{q}\cos\left(\frac{2\pi\,\Delta\phi_1}{\Phi_0}\right) - \frac{\hbar I_{c_2}}{q}\cos\left(\frac{2\pi(\Delta\phi_1 + \Phi_3)}{\Phi_0}\right) \tag{10}$$

where we have dropped any unnecessary constants and the (weakly coupled) harmonic oscillator terms. This Hamiltonian is then the simplest which describes both the transverse field excitations of the bulk ring and localized charge states around the point contact. We have a transverse field commutator ($[Q'_3, \Phi_3] = \mathrm{i}\hbar$) between two continuous operators, and a longitudinal field commutator ($[q_2, \Delta\phi_1] = -\mathrm{i}\hbar$) between a number operator ($\times q$) and an angular operator (with period $= \Phi_0$).

Next, we perform a time-dependent unitary transform to obtain a new Hamiltonian [13, 28]

$$H' = U^\dagger H U - \mathrm{i}\hbar U^\dagger \frac{\partial U}{\partial t}$$

where $U = \exp(\mathrm{i}\Phi_{ex}(t)Q'_3/\hbar)$, giving a Hamiltonian which contains an applied Faraday law voltage $V_{ex} = -\partial\Phi_{ex}/\partial t$ and an external flux $\Phi_{ex}(t)$,

$$H' = \frac{q_2^2}{2C'_{22}} + \frac{q_2 Q'_3}{2C'_{23}} + \frac{(Q'_3)^2}{2C'_{33}} - q'_3 V_{ex} + \frac{\Phi_3^2}{2\Lambda_3} - \frac{\hbar I_{c_1}}{q}\cos\left(\frac{2\pi\,\Delta\phi_1}{\Phi_0}\right) - \frac{\hbar I_{c_2}}{q}\cos\left(\frac{2\pi(\Delta\phi_1 + \Phi_3 + \Phi_{ex}(t))}{\Phi_0}\right). \tag{11}$$

We note that, although this Hamiltonian is similar to one given in [13] with only one cosine term, the additional tunnelling term cannot be removed by letting I_{c_1} or I_{c_2} tend to zero, since this would also effect the self-capacitance of the segment, $C^{(sc)} \to 0$. Taking the critical current for either segment to be zero would lead to an additional constraint, $q_1 = 0$ or $q_2 = 0$. This effectively removes the additional degree of freedom, leaving a Hamiltonian which is again periodic in applied magnetic flux (although the periodicity will be $(1 + [C_{22}^{-1} + (2C_{23})^{-1}]/[C_{33}^{-1} + (2C_{23})^{-1}])\Phi_0$, rather than the usual Φ_0).

Taking this Hamiltonian, we begin by considering two limiting cases: $\hbar I_{c_1}/q \gg q^2/C'_{22}$, and $\hbar I_{c_1}/q \ll q^2/C'_{22}$. (Without loss of generality, we take I_{c_2} to be smaller than I_{c_1}, since we can interchange the role of each term by another unitary transformation $U' = \exp(\mathrm{i}(\Phi_{ex}(t) + \Phi_3)q_2/\hbar)$.)

The first case, $\hbar I_{c_1}/q \gg q^2/C'_{22}$, will tend to give low-energy states which are localized in phase. In this limit, the segment is no longer weak and will act more like a bulk superconductor. Strictly speaking, we should use (9) rather than (11). In either case, where $\Delta\phi_1$ is very well defined (around zero or Φ_1, respectively) the model reduces to the well known rf SQUID Hamiltonian [26], which has been extensively discussed elsewhere [11, 12]. The time independent Schrödinger equation for this macroscopic Hamiltonian gives energy levels ($E_\kappa(\Phi_{ex})$) which are Φ_0 periodic when a static (or quasi-static) magnetic flux is applied. These flux dependent levels can be probed experimentally through their κ dependent magnetic susceptibilities $\chi_\kappa(\Phi_{ex})$ [12], where

$$\chi_\kappa(\Phi_{ex}) = -\Lambda_3 \frac{\partial^2 E_\kappa(\Phi_{ex})}{\partial \Phi_{ex}^2}$$

is a dimensionless quantity. In these well defined phase states, the quantum fluctuations in the charge (q_2) will be too large to discern any voltage periodic effects from the quantization of real charge in units of $q = 2e$. (Localized charge states will exist, but they will have very high energies.) The ring will show voltage dependent behaviour, but only to the extent that the applied voltage can be seen as a source of time dependent magnetic flux. For slowly varying (adiabatic) magnetic fluxes (i.e. low voltages) the system will remain in one energy state, sweeping out the periodic response. For higher voltages, the magnetic flux will be non-adiabatic and the evolution will be more complicated.

For $\hbar I_{c_1}/q \ll q^2/C'_{22}$, the phase will be less well defined, and the low-lying energy states will tend to be more strongly localized in real charge. As the system becomes delocalized in phase, such that $\langle \Delta\phi_1^2 \rangle \gtrsim \Phi_0^2$, we would expect the system to lose sensitivity to magnetic flux. Any physical quantity, related to an expectation value, must now be averaged over more than one period of Φ_0. The Hamiltonian will still be periodic in static magnetic flux, with the same periodicity, but the amplitude of the response will reduce, giving resultant energy levels which are flat in Φ_{ex}. The κ dependent magnetic susceptibility will therefore vanish (see figure 2).

This is the regime in which we would expect to see voltage periodic behaviour, and indeed this is the case. Rather than applying a static magnetic flux, we now apply a linearly ramped flux $\Phi_{ex}(t) = -V_{ex}t + \Phi_e^{(dc)}$ (where $\Phi_{ex}^{(dc)}$ is a static (dc) offset). This gives a static (Faraday law) voltage, which can be used to probe the electric susceptibility of the ring [29],

$$\chi_\kappa^{(E)}(V_{ex}) = \frac{1}{C}\frac{\partial^2 E_\kappa(V_{ex})}{\partial V_{ex}^2}$$

which is κ dependent and periodic in the applied voltage (period $= 2e/C$).

In the presence of the static voltage, we can rewrite the Hamiltonian (using $Q''_3 = Q'_3 - C'_{22}V_{ex}$).

$$\begin{aligned} H'(t) = {} & \left[\frac{(q_2 - CV_{ex})^2}{2C'_{22}} - \frac{\hbar I_{c_1}}{q}\cos\left(\frac{2\pi\,\Delta\phi_1}{\Phi_0}\right)\right] \\ & + \frac{(Q''_3)^2}{2C'_{33}} + \frac{\Phi_3^2}{2\Lambda_3} + \frac{q_2 Q''_3}{2C'_{23}} - \left[\frac{C'_{22}}{2}\left(\frac{C'_{33}}{2C'_{23}}\right)^2 + \frac{C'_{33}}{2}\right]V_{ex}^2 \\ & - \frac{\hbar I_{c_2}}{q}\cos\left(\frac{2\pi(\Delta\phi_1 + \Phi_3 - V_{ex}t + \Phi_{ex}^{(dc)})}{\Phi_0}\right) = H'_0 + H'_I(t) \end{aligned} \tag{12}$$

where $C = (C'_{22}C'_{33}/2C'_{23})$. Using this form of the Hamiltonian, we can calculate the electric susceptibility for the lowest few energy levels. (The fact that the Hamiltonian is time dependent complicates the situation. However, given that I_{c_2} is comparatively small, we can treat the time dependent term $H'_I(t)$ as a perturbation [28]. Provided that the frequency $\omega = (2\pi V_{ex}/\Phi_0)$ does not correspond to a transition between energy levels (i.e. it is far off resonance), we can average over one period of the time dependent term.) As an example, we consider a macroscopic superconducting ring and take the longitudinal (Coulomb) capacitances to be $C^{(l)} \sim 10^{-12}$ F, with a transverse capacitance $C_3^{(t)} \sim 10^{-15}$ F and inductance $\Lambda_3 = 3 \times 10^{-10}$ H [26]. The condensate self-capacitances $C_1^{(sc)}$ and $C_2^{(sc)}$ will be proportional to the values of the critical currents ($I_{c_2} < I_{c_1}$). Using, $v_F = 10^6$ m s^{-1}, $(\Delta x_1) = (\Delta x_2) = 1$ μm, with $I_{c_2} = 0.2$ μA and $I_{c_1} = 2$ μA [15], we obtain values for the self-capacitances $C_1^{(sc)} = 1 \times 10^{-16}$ F and $C_2^{(sc)} = 1 \times 10^{-17}$ F. This gives approximate values for $C'_{33} \simeq C_3^{(t)} \sim 10^{-15}$ F, $C'_{23} \simeq C_3^{(t)}/2 \sim 5 \times 10^{-16}$ F and

$$C'_{22} \simeq C \simeq \left(\frac{1}{C_1^{(sc)}} + \frac{1}{C_2^{(sc)}}\right)^{-1} \simeq 10^{-17}\ \text{F}.$$

Using these parameter values, we can calculate the magnetic and electric susceptibilities of a macroscopic ring (using a basis consisting of harmonic oscillator states for the transverse field and discrete charge states for the longitudinal field). In figure 2, we show the ground state ($\kappa = 0$) magnetic susceptibility for $I_{c_1} = 2$, 8 and 20 μA. We notice that the flux periodicity of the ring does remain, but that the response is very much suppressed as I_{c_1} is reduced and the low-lying energy states become delocalized in phase. For $I_{c_1} = 2$ μA, we also calculate the voltage periodic electric susceptibility of the lowest three energy states ($\kappa = 0, 1, 2$) (see figure 3). In this case, the very weak response to an applied magnetic flux (shown in figure 2) is in marked contrast to the very strong features shown in the electric susceptibility. The similarity between the electric susceptibilities of the first few energy states is due to the weak coupling between the transverse field and the longitudinal field for the parameter values used in the calculation. The lowest-energy excitations correspond (approximately) to excited states of the transverse field, and the ground state of the longitudinal field. For other parameter values, where the coupling between the fields is stronger, the electric susceptibilities will exhibit much more structure.

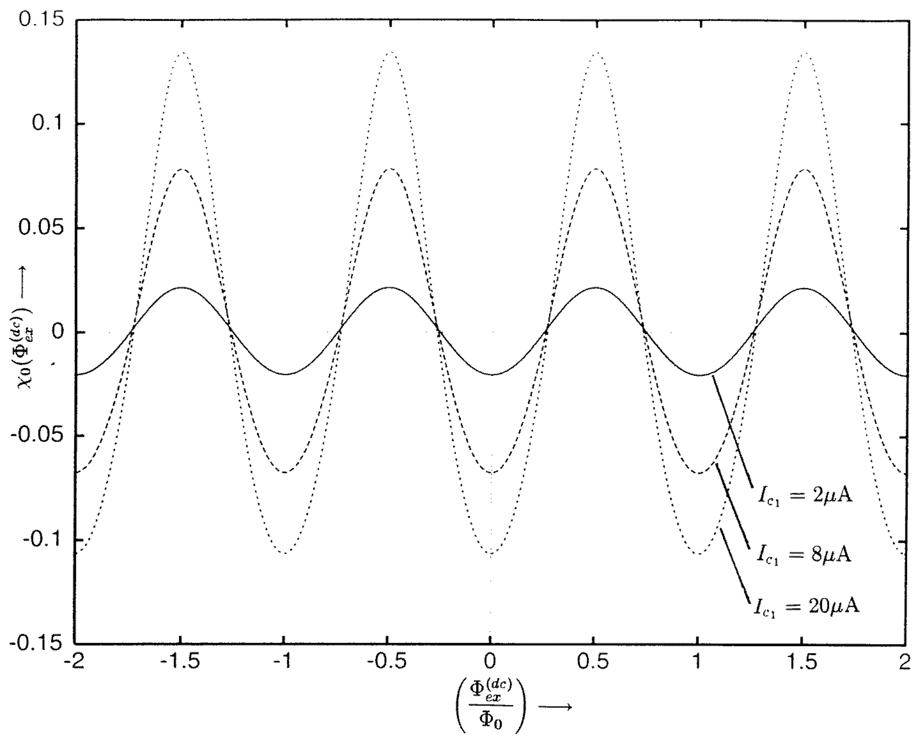

Figure 2. Ground state ($\kappa = 0$) magnetic susceptibility against static magnetic flux $\Phi_{ex}^{(dc)}$ for a macroscopic ring with $I_{c_1} = 2$, 8 and 20 μA (other parameter values are given in the text).

It is clear from figures 2 and 3 that our example ring can exhibit both voltage and flux dependent behaviour. This result is a general feature of the symmetry broken model. However, we also see that where we have very sharp features in the electric susceptibility, we obtain very weak features in the magnetic flux response. To this extent, the voltage periodic charge mode and the flux periodic flux mode can be thought of as conjugate modes of operation [4]. One is associated with localized charge states around the weak link and the

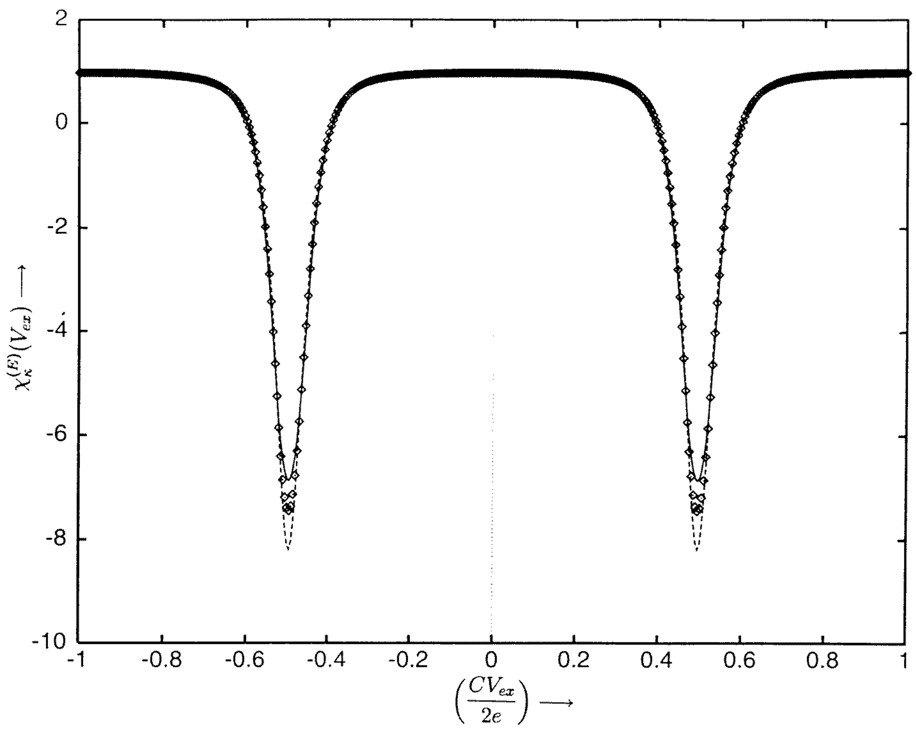

Figure 3. Electric susceptibilities against a static applied voltage V_{ex}, calculated for the first three energy levels ($\kappa = 0, 1, 2$) with $I_{c_1} = 2\ \mu$A ($\kappa = 0$, dashed line; $\kappa = 1$, diamonds; $\kappa = 2$, solid line; all other parameter values are as given in the text).

other with localized phase states. However, the situation is not quite as simple as previously stated [4], since the localized charge states found in the charge mode are physical excitations of the longitudinal electromagnetic field [13] (due to the broken symmetry [15]) and the flux mode is associated with the transverse field states.

The other interesting result we obtain from this model is that the flux and charge modes are limiting cases: $\hbar I_{c_1}/q \gg q^2/C'_{22}$, and $\hbar I_{c_1}/q \ll q^2/C'_{22}$ respectively. Between these two extremes, we should obtain a more general type of behaviour, where the superconducting ring is sensitive to static magnetic flux, but still retains some of the voltage periodic behaviour found in the charge mode. If this intermediate regime can be probed experimentally, it should provide a rich source of new structure for quantum mechanical superconducting circuits, and may provide a valuable insight into the interplay between microscopic quantum effects (localized charge states) and macroscopic quantum effects (macroscopic flux states) in superconductors.

So far, we have only discussed the behaviour of the superconducting condensate. We have not mentioned the quasi-particle/normal fluid degrees of freedom, which become important once above the superconducting energy gap [21]. The inclusion of these effects is beyond the scope of this paper, but their effects have been extensively discussed elsewhere [8, 30]. In addition, any experimental system must also allow for other (extraneous) environmental degrees of freedom. It is well known that different environmental systems can effect the evolution of quantum mechanical circuits in different ways [31]. One effect can be to shift the effective capacitance of a circuit [8, 31] away from its actual (bare) value. Since the behaviour that we have been discussing is dependent on the capacitive energy of the weak link and the capacitive energy of the transverse electric field in the ring, we would expect the electromagnetic environment to play a crucial role in determining the behaviour of a real superconducting circuit. By introducing, or removing, additional environmental degrees of freedom it may be possible to move from one regime to another by very carefully controlling the capacitive energy of the weak link [16].

In conclusion, we have taken a quantum electrodynamic model with broken gauge symmetry and constructed an equivalent circuit model for the superconducting condensate in a weak-link ring. This model describes the transverse field states (which occur in normal conductors) and the physical states of the longitudinal field (which do not). We have then taken a specific example which can be used to describe the transverse excitations of the bulk ring and the localized charge states of the weak link. The resultant Hamiltonian was then used to obtain flux periodic magnetic susceptibilities and voltage periodic electric susceptibilities, as limiting cases of a more general phenomenon. The flux dependent behaviour was found to correspond to transverse field excitations in the presence of a localized phase, whilst the voltage periodic behaviour was associated with localized charge states of the longitudinal field. That this model can be used to generate the two modes of operation that can be found in experimental weak-link rings [4] is very important; but the fact that a more general regime (exhibiting some 'charge mode' and some 'flux mode' characteristics) may be realizable in experiment is potentially even more exciting.

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